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Discrete Mathematics 306 (2006) 87–98
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Spanning trails containing given edges

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Received 4 March 2004; received in revised form 6 October 2005; accepted 25 October 2005
Available online 6 January 2006

Abstract

A graph G is Eulerian-connected if for any u and v in $V(G)$, G has a spanning (u, v)-trail. A graph G is edge-Eulerian-connected if for any e' and e'' in $E(G)$, G has a spanning (e', e'')-trail. For an integer $r \geqslant 0$, a graph is called r-Eulerian-connected if for any $X \subseteq E(G)$ with $|X| \leqslant r$, and for any $u, v \in V(G)$, G has a spanning (u, v)-trail T such that $X \subseteq E(T)$. The r-edge-Eulerian-connectivity of a graph can be defined similarly. Let $\theta(r)$ be the minimum value of k such that every k-edge-connected graph is r-Eulerian-connected. Catlin proved that $\theta(0) = 4$. We shall show that $\theta(r) = 4$ for $0 \leqslant r \leqslant 2$, and $\theta(r) = r + 1$ for $r \geqslant 3$. Results on r-edge-Eulerian connectivity are also discussed.

Keywords: Connectivity; Spanning trails; Supereulerian graphs; Collapsible graphs

1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. A graph G is *Hamiltonian-connected* if for every pair of vertices u, v of G, there is a Hamiltonian (u, v)-path in G. For a graph G, a trail is a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_{k-1}, v_{k-1}, e_k, v_k$ such that all the e_i's are distinct and $e_i = v_{i-1}v_i$ for all i. Let $e', e'' \in E(G)$. A trail in G whose first edge is e' and whose last edge is e'' is called an (e', e'')-trail. For $u, v \in V(G)$, a (u, v)-trail of G is a trail in G whose origin is u and whose terminus is v. A trail H is called a *dominating trail* of G if every edge of G is incident with at least one vertex of H in G. A trail H is called a *spanning trail* if $V(H) = V(G)$. If $u = v$, then a (u, v)-trail in G is a closed trail, which is also called a *Eulerian subgraph* of G. A graph is called *supereulerian* if it has a spanning closed trail. The collection of all supereulerian graphs is denoted by $\mathcal{SL}$.

A graph G is *Eulerian-connected* if for any u, v in $V(G)$ (including the case $u = v$), G has a spanning (u, v)-trail. A graph is called r-Eulerian-connected if for any $X \subseteq E(G)$ with $|X| \leqslant r$, and for any $u, v \in V(G)$, G has a spanning

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.10.022

(u, v)-trail T such that $X \subseteq E(T)$. For an integer $r \geqslant 0$, the collection of all r-Eulerian-connected graphs is denoted by $\mathscr{EL}(r)$. Obviously, $\mathscr{EL}(r) \subseteq \mathscr{SL}$ for all $r \geqslant 0$.

A graph G is *edge-Eulerian-connected* if for any e', e'' in $E(G)$, G has a spanning (e', e'')-trail. A graph is called r-edge-Eulerian-connected if for any $X \subseteq E(G)$ with $|X| \leqslant r$ and for any $e', e'' \in E(G)$, G has a spanning (e', e'')-trail T such that $X \subseteq E(T)$. For an integer $r \geqslant 0$, the collection of all r-edge-Eulerian-connected graphs is denoted by $\mathscr{EE}(r)$.

Many studies have been done on Eulerian graphs (see [7]). For the literature on the subject of supereulerian graphs, see surveys [3,6]. Harary and Nash-Williams [9] demonstrated the relationship between Eulerian subgraphs and Hamiltonian cycles in the line graph of G. Zhan [14] studied (e', e'')-trails of a graph G for the Hamiltonian connectivity of the line graph of G. In the study of spanning trails of graphs [2], Catlin introduced the concept of *collapsible graphs*. For a graph G, let $\mathrm{O}(G)$ be the set of odd degree vertices of G and let R be an even subset of $V(G)$. A subgraph H_R of G is called a *spanning R-trail* if H_R is a spanning connected subgraph such that $\mathrm{O}(H_R) = R$. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, G has a spanning R-trail. We will regard an empty set as an even subset and K_1 as both collapsible and supereulerian. The collection of all collapsible graphs is denoted by $\mathscr{CL}$. By the definition of collapsible graphs, we have:

Proposition A. *Let G be a collapsible graph. Then each of the following holds*

(i) *G is supereulerian.*
(ii) *G is Eulerian-connected.*

Proof. For any vertices $u, v \in V(G)$. Let $R = \emptyset$ if $u = v$, or $R = \{u, v\}$ if $u \neq v$. Since G is collapsible, it has a spanning subgraph H_R such that $\mathrm{O}(H_R) = R$. Therefore, H_R is a spanning Eulerian subgraph of G if $R = \emptyset$, or H_R is a (u, v)-spanning trail of G. $\square$

Let $X \subseteq E(G)$ and let R be an even subset of $V(G)$. A spanning R-trail H_R of G such that $X \subseteq E(H_R)$ is called *a spanning* (R, X)*-trail*, and denoted by $H_R(X)$. A graph is called strongly r-Eulerian-connected if for any $X \subseteq E(G)$ with $|X| \leqslant r$ and for any even subset $R \subseteq V(G)$, G has a spanning R-trail H_R such that $X \subseteq E(H_R)$ (i.e. G has a $H_R(X)$). The collection of all strongly r-Eulerian-connected graphs is denoted by $\mathscr{SE}(r)$.

For an integer r, define $\mathscr{L}(r)$ to be the family of graphs such that $G \in \mathscr{L}(r)$ if and only if for any subset $X \subseteq E(G)$ with $|X| \leqslant r$, G has an spanning Eulerian subgraph H such that $X \subseteq E(H)$. Define $f(r)$ to be the minimum value of k such that every k-edge-connected graph G is in $\mathscr{L}(r)$. In [12], Lai found $f(r)$ for all the values of r (see Corollary 3.6). Let $\theta(r)$ be the minimum value of k such that every k-edge-connected graph is in $\mathscr{EL}(r)$ and let $\psi(r)$ be the minimum value of k such that every k-edge-connected graph is in $\mathscr{SE}(r)$. Since $\mathscr{SE}(r) \subseteq \mathscr{EL}(r) \subseteq \mathscr{L}(r)$,

$$f(r) \leqslant \theta(r) \leqslant \psi(r). \tag{1}$$

Let $\xi(r)$ be the minimum value of k such that every k-edge-connected graph is in $\mathscr{EE}(r)$. In this paper, we will determine the values of $\theta(r)$, $\psi(r)$, and $\xi(r)$ for all $r \geqslant 0$.

In the next section, we will present Catlin's reduction method and some preliminary results which are needed in our proofs. Our main results are in Sections 3 and 4. We will present our results on r-Eulerian-connected graphs, and give the values of $\theta(r)$ and $\psi(r)$ for all $r \geqslant 0$. Section 4 contains results on the r-edge-Eulerian connected graphs.

2. Catlin's reduction method and preliminary results

Let H be a connected subgraph of G. The contraction G/H is obtained from G by contracting each edge of H and deleting the resulting loops. In [2], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The *reduction* of G is obtained from G by contracting each of H_i into a vertex v_i for all i, and is denoted by G'. Each H_i is called a preimage of v_i in G, and v_i is called the contraction image of H_i in G'. A vertex v in G' is called a *trivial contraction* if its preimage in G is K_1. A graph G is reduced if G is the reduction of some graph. Let $F(G)$ be the minimum number of edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees.

Theorem 2.1 (*Catlin [2]*). *Let G be a graph, and let G' be the reduction of G. Each of the following holds.*

(i) *G is supereulerian if and only if G' is supereulerian.*
(ii) *G is collapsible if and only if $G' \cong K_1$*
(iii) $|E(G')| + F(G') = 2|V(G')| - 2$.

In [10], Jaeger proved that a graph with two edge-disjoint spanning trees is supereulerian. In [2], Catlin proved that if G has two edge-disjoint spanning trees, then G is collapsible. It is well known now that a $2k$-edge-connected graph has k edge-disjoint spanning trees [8,11,13]. Thus, we have:

Theorem 2.2. *If G is* 4*-edge-connected, then G is collapsible.*

In [4], Catlin proved:

Theorem 2.3 (*Catlin [4]*). *Let G be a graph and let $k \geqslant 1$ be an integer. The following are equivalent:*

(i) *G is $2k$-edge-connected;*
(ii) *For any $X \subseteq E(G)$ with $|X| \leqslant k$, $G - X$ has k edge-disjoint spanning trees.*

Corollary 2.4 (*Catlin [4]*). *Let G be a graph and let $k \geqslant 1$ be an integer. The following are equivalent:*

(i) *G is $(2k+1)$-edge-connected;*
(ii) *For any $X \subseteq E(G)$ with $|X| \leqslant k+1$, $G - X$* has *k-edge-disjoint spanning trees.*

The following theorems will be needed in our proofs.

Theorem 2.5 (*Catlin et al. [5]*). *Let G be a connected graph. If $F(G) \leqslant 2$, then either G is collapsible, or the reduction of G is in $\{K_2, K_{2,t} : t \geqslant 1\}$.*

Let e be an edge in G. Edge e is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that path of length 2 is called subdividing e. Let G be a graph and let $X \subseteq E(G)$. Let G_X be the graph obtained from G by subdividing each edge in X. Then $V(G_X) = V(G) \cup \{v(e) \text{ for each } e \in X\}$.

Lemma 2.6. *Let $k \geqslant 2$ be an integer. Let G be a connected graph and let $X \subseteq E(G)$. Let R be an even subset of $V(G)$. Then each of the following holds*

(i) *G has a spanning (R, X)-trail $H_R(X)$ if and only if G_X has a spanning R-trail. In particular, G has a spanning closed trail H such that $X \subseteq E(H)$ if and only if G_X is supereulerian.*
(ii) *If G_X is collapsible, then G_X has a spanning R-trail.*
(iii) *Let $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$. Then $F(G_X) \leqslant F((G - X_1)_{X_2})$.*
(iv) *If G has k edge-disjoint spanning trees, then for any $X \subseteq E(G)$ with $|X| \leqslant 2k-2$, $F(G_X) \leqslant 2$.*

Proof. (i) and (ii) follow from the definitions of collapsibility and G_X.

(iii) Let $p = F((G - X_1)_{X_2})$. Let E_p be the p edge set such that $(G - X_1)_{X_2} + E_p$ has 2-edge-disjoint spanning trees (T_1 and T_2). Let $X_1 = \{e_1, e_2, \ldots, e_s\}$ and each $e_i = u_i v_i$ $(1 \leqslant i \leqslant s)$. By the definition of G_X, we know that G_X can be obtained from $(G - X_1)_{X_2}$ by joining each pair of u_i and v_i by a path $P_i = u_i v(e_i) v_i$ where $v(e_i)$ is a new vertex. Therefore, $T_1 + \bigcup_{i=1}^{s} \{u_i v(e_i)\}$ and $T_2 + \bigcup_{i=1}^{s} \{v(e_i) v_i\}$ are two edge-disjoint spanning trees in $G_X + E_p$, and so $F(G_X) \leqslant p = F((G - X_1)_{X_2})$.

(iv) Let $T_1, T_2, \ldots, T_k$ be k edge-disjoint spanning trees of G. Without lost of generality, we may assume that

$$|X \cap E(T_1)| \leqslant |X \cap E(T_2)| \leqslant \cdots \leqslant |X \cap E(T_k)|. \tag{2}$$

Since $k \geqslant 2$, $|X| \leqslant 2k-2$, T_i's are edge-disjoint, and by (2),

$$|X \cap E(T_1)| + |X \cap E(T_2)| \leqslant 2. \tag{3}$$

Let $X = \{e_1, e_2, \dots, e_p\}$ where $p \leqslant 2k-2$, and let $e_i = u_i v_i$ for all $1 \leqslant i \leqslant p$. Since G_X is the graph obtained from G by subdividing e_i $(1 \leqslant i \leqslant p)$, $V(G_X) = V(G) \cup \{v(e_i) : 1 \leqslant i \leqslant p\}$, and $E(G_X) = (E(G) - X) \cup \{u_i v(e_i), v(e_i)v_i : 1 \leqslant i \leqslant p\}$.

Case 1. $|X \cap E(T_1)| + |X \cap E(T_2)| = 0$.

Then $T_1 + \bigcup_{i=1}^{p} \{u_i v(e_i)\}$ and $T_2 + \bigcup_{i=1}^{p} \{v(e_i)v_i\}$ are two edge-disjoint spanning trees in G_X and so $F(G_X) = 0 \leqslant 2$.

Case 2. $|X \cap E(T_1)| + |X \cap E(T_2)| = 1$.

By (2) and (3), $|X \cap E(T_1)| = 0$ and $|X \cap E(T_2)| = 1$. Let $e_2 = u_2 v_2$ be the edge in $X \cap E(T_2)$. Then $T_2' = T_2 - e_2 + \{u_2 v(e_2), v(e_2)v_2\} \bigcup_{i \neq 2}^{p} \{v(e_i)v_i\}$ is a spanning tree in G_X. To obtain another spanning tree which covers $v(e_2)$, we can add an edge $e' = u_1 v(e_2)$ to G_X. Then $T_1' = T_1 + \{e'\} \bigcup_{i \neq 2}^{p} \{u_i v(e_i)\}$ is a spanning tree in $G_X + e'$. Therefore, T_1' and T_2' are two edge-disjoint spanning trees in $G_X + e'$. This shows that $F(G_X) = 1 \leqslant 2$.

Case 3. $|X \cap E(T_1)| + |X \cap E(T_2)| = 2$.

By (2) and (3), either $|X \cap E(T_1)| = |X \cap E(T_2)| = 1$, or $|X \cap E(T_1)| = 0$ and $|X \cap E(T_2)| = 2$. We prove $F(G_X) \leqslant 2$ for the case $|X \cap E(T_1)| = |X \cap E(T_2)| = 1$ here. The case $|X \cap E(T_1)| = 0$ and $|X \cap E(T_2)| = 2$ can be proved similarly.

Let $e_1 \in X \cap E(T_1)$ and $e_2 \in X \cap E(T_2)$. Then $T_1' = T_1 - e_1 + \{u_1 v(e_1), v(e_1)v_1\} \bigcup_{i=3}^{p} \{u_i v(e_i)\}$ is a tree containing $V(G_X) - v(e_2)$, and $T_2' = T_2 - e_2 + \{u_2 v(e_2), v(e_2)v_2\} \bigcup_{i=3}^{p} \{v(e_i)v_i\}$ is a tree containing $V(G_X) - v(e_1)$. Therefore, adding two new edges $e' = u_1 v(e_2)$ and $e'' = v(e_1)v_2$ to G_X, we have two edge-disjoint spanning trees $T_1' + e'$ and $T_2' + e''$ in $G_X + \{e', e''\}$. This shows that $F(G_X) \leqslant 2$. The proof is complete. □

Lemma 2.7. *Let G be a graph with $\kappa'(G) \geqslant 3$, and let $X \subseteq E(G)$. Let G_X be the graph obtained from G by subdividing each edge in X. If the reduction of G_X is $K_{2,t}$, then each of the following holds.*

(i) *Every degree 2 vertex in G_X' is a vertex obtained by subdividing an edge in X.*
(ii) *$|X| \geqslant t \geqslant \kappa'(G)$, and X is an edge cut of G.*
(iii) *There is a subset $X_1 \subseteq X$ with $t = |X_1|$ such that each path between the two vertices of degree t in $K_{2,t}$ is obtained by subdividing an edge in X_1. Furthermore, $G_X - X_1$ has only two collapsible components (say H_1 and H_2) such that $V(G_X) = V(H_1) \cup V(H_2) \bigcup_{e \in E_1} \{v(e)\}$, and $G_X' = K_{2,t}$ is obtained by contracting H_1 and H_2 (i.e. $G_X' = (G_X/H_1)/H_2 = K_{2,t}$).*

Proof. Let $E(G_X') = E(K_{2,t}) = \{uw_i, w_i v\}$ $(1 \leqslant i \leqslant t)$ where each w_i is a degree 2 vertex in G_X'. Note that w_i is a trivial contraction, and (i) holds. Otherwise the two edges incident with w_i will form an edge-cut of G, contrary to that $\kappa'(G) \geqslant 3$. Hence, each path $uw_i v$ is obtained by subdividing an edge in X and so $t \leqslant |X|$.

Let $E' = \{uw_i : 1 \leqslant i \leqslant t\}$. Then E' is an edge-cut of G_X'. Since each path $uw_i v$ in G_X is obtained by subdividing an edge $e \in X \subseteq E(G)$, we have an edge set $X_1 \subseteq X$ such that each edge in X_1 corresponding to a path $uw_i v$ in G_X, and $|X_1| = |E'| = t$. Therefore, X_1 is an edge cut in G. Since $X_1 \subseteq X$, X is an edge-cut of G and $|X| \geqslant |E'| = t \geqslant \kappa'(G)$.

Note $V(G_X') = \{u, v, w_i : 1 \leqslant i \leqslant t\}$ where $d(u) = d(v) = t$. Let H_1 be the preimage of u, and let H_2 be the preimage of v. Therefore, G_X' is obtained by subdividing each edge in X_1, and then contracting H_1 and H_2, respectively. Statement (iii) is proved. □

Lemma 2.8. *Let G be an r-edge-connected graph $(r \geqslant 4)$. Let $X \subseteq E(G)$. Let G_X be the graph obtained from G by subdividing each edge in X. Let G_X' be the reduction of G_X and let V_r be the set of vertices of degree less than r in G_X'. Let $D_i = \{v \in V(G_X') : d(v) = i\}$ $(i \geqslant 2)$. If $F(G_X') \geqslant 3$, then each of the following holds*:

(i) *each vertex in V_r has degree 2 (i.e. $V_r = D_2$) and $|V_r| \leqslant |X|$.*
(ii) *$(r-4)|V(G_X')| + 10 \leqslant (r-2)|V_r| \leqslant (r-2)|X|$.*
(iii) *$10 + (r-4)|D_r| + (r-3)|D_{r+1}| + \cdots + \leqslant 2|V_r| \leqslant 2|X|$.*

Proof. Since the degree of each vertex u in V_r is less than r, u must be a trivial contraction in G_X'. Otherwise, the edges incident with u will form an edge cut with size less than r, contrary to $\kappa'(G) \geqslant r$. Therefore, $V_r \subseteq V(G_X) - V(G)$, a subset of the vertices obtained in the process of subdividing each edge in X. Thus each vertex in V_r has degree 2 and

$$|V_r| \leqslant |X|. \tag{4}$$

Let $c = |V(G'_X)|$. Since $F(G'_X) \geqslant 3$, by (iii) of Theorem 2.1,

$$|E(G'_X)| = 2|V(G'_X)| - 2 - F(G'_X) \leqslant 2c - 5.$$

Hence,

$$\sum_{v \in V(G'_X)} d(v) = 2|E(G'_X)| \leqslant 4c - 10. \tag{5}$$

Since $\kappa'(G_X) \geqslant 2$, $\delta(G'_X) \geqslant 2$. Then by (5)

$$2|V_r| + r(c - |V_r|) \leqslant 2|V_r| + \sum_{v \notin V_r} d(v) = \sum_{v \in V(G'_X)} d(v) = 2|E(G'_X)| \leqslant 4c - 10. \tag{6}$$

By (4), (6), and $c = |V(G'_X)|$,

$$(r - 4)|V(G'_X)| + 10 \leqslant (r - 2)|V_r| \leqslant (r - 2)|X|. \tag{7}$$

By (6), and $V(G'_X) = V_r \bigcup_{i=r} D_i$,

$$2|V_r| + r|D_r| + (r + 1)|D_{r+1}| + \cdots \leqslant 4(|V_r| + |D_r| + |D_{r+1}| + \cdots) - 10.$$

Hence,

$$10 + (r - 4)|D_r| + (r - 3)|D_{r+1}| + \cdots \leqslant 2|V_r| \leqslant 2|X|. \qquad \square$$

Lemma 2.9. *Let G be a graph and let $e_1, e_2 \in E(G)$ and let $X \subseteq E(G)$. Let $X_0 = X \cup \{e_1, e_2\}$. Let G_{X_0} be the graph obtained from G by subdividing each edge in X_0. Let $v(e_1)$ and $v(e_2)$ be the two vertices subdividing e_1* and e_2, *respectively. Then*

(i) *If G_{X_0} has a spanning $(v(e_1), v(e_2))$-trail, then G has a spanning (e_1, e_2)-trail containing X.*
(ii) *If G_{X_0} is collapsible, then G has a spanning (e_1, e_2)-trail containing X.*

Proof. Follows from the definitions of collapsibility and G_{X_0}. $\square$

3. The *r*-Eulerian-connected graphs

The Petersen graph and many other 3-edge-connected graphs have no spanning closed trails. Thus, for any $r \geqslant 0$, $\psi(r) \geqslant \theta(r) \geqslant 4$. By Theorem 2.2, we know that $\psi(0) = \theta(0) = 4$. The following example shows that for $r \geqslant 3$, $\psi(r) \geqslant \theta(r) \geqslant r + 1$.

Example 1. Let $r \geqslant 3$ be an integer, and let n and m be two integers such that $n \geqslant r + 1$ and $m \geqslant r + 1$. Let $G_1 = K_n$ with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$, and let $G_2 = K_m$ with $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Define the graph G to be the graph obtained from G_1 and G_2 by connecting G_1 and G_2 with the new edge set $X = \{e_1, e_2, \ldots, e_r\}$ where $e_i = u_i v_i$ for all $i = 1, 2, \ldots, r$. Then G is an r-edge-connected graph. If r is even, then we choose u from G_1, and v from G_2. If r is an odd integer, then we choose u and v both from G_1. Then G has no spanning (u, v)-trails containing all the edges of X. This example also shows that G has no spanning (e', e'')-trails containing all the edges of X for some pair of $e', e'' \in E(G)$. See Fig. 1 below for the case $r = 4$ where $X = \{e_1, e_2, e_3, e_4\}$ and $G_1 \cong G_2 \cong K_5$. This shows that $\psi(r) \geqslant \theta(r) \geqslant r + 1$. In the following, we will show that $\psi(r) = \theta(r) = r + 1$.

This example suggests the following necessary condition for r Eulerian-connected graphs, and the lower bounds for $\psi(r)$, $\theta(r)$ and $\xi(r)$.

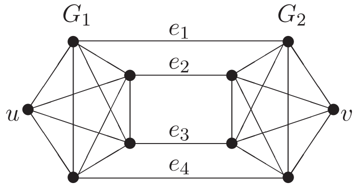

Fig. 1.

Theorem 3.0. *Let $r \geqslant 3$. Then $\psi(r) \geqslant \theta(r) \geqslant r+1$ and $\xi(r) \geqslant r+1$. Furthermore, if G is an r-Eulerian-connected graph, then G is $(r+1)$-edge-connected.*

Proof. By way of contradiction, suppose that the edge-connectivity of G is $k \leqslant r$. Let X be an edge cut with $|X| = k$ and let H_1 and H_2 be two components of $G - X$. If $|X| = k$ is even, we can choose a vertex u from H_1 and a vertex v from H_2. Then G has no spanning (u, v) trail that contains X, a contradiction. If $|X| = k$ is odd, then we can choose a vertex u from H_1. Since X has odd number of edges, G does not have a closed trail that starts and ends at u containing X, a contradiction again. □

For a real number x, let $\lfloor x \rfloor$ be the largest integer that is less than or equal to x.

Theorem 3.1. *Let $r \geqslant 4$ be an integer and let $k = \lfloor \frac{r}{2} \rfloor$. Let G be an r-edge-connected graph and let $X \subseteq E(G)$ with $|X| \leqslant r+k-2$.* it Then one of the following holds:

(i) *G_X is collapsible, or*
(ii) *X is an edge cut of G and $|X| \geqslant r$.*

Proof. Let $X \subseteq E(G)$ with $|X| \leqslant r+k-2$. Define G_X as before and assume that G_X is not collapsible. We will show that the reduction G'_X is $K_{2,t}$ with $t \geqslant 2$ first. Consider the following two cases:

Case 1. r is even. Then $r = 2k$, and $|X| \leqslant 3k-2$.

Since $|X| \leqslant 3k-2$, we can choose a subset X_1 of X and let $X_2 = X - X_1$, such that $|X_1| \leqslant k$ and $|X_2| \leqslant 2k-2$. By Theorem 2.3, $G - X_1$ has k-edge-disjointed spanning trees. Then by Lemma 2.6(iv), $F((G-X_1)_{X_2}) \leqslant 2$. By Lemma 2.6(iii), $F(G_X) \leqslant F((G-X_1)_{X_2}) \leqslant 2$. Since G_X is not collapsible, by Theorem 2.5, $G'_X \in \{K_2, K_{2,t}\}$ $(t \geqslant 1)$. Since G is r-edge-connected $(r \geqslant 4)$, G_X is 2-edge-connected. Therefore, $G'_X = K_{2,t}$ $(t \geqslant 2)$.

Case 2. r is odd. Then $r = 2k+1$ and $|X| \leqslant 3k-1$.

Let X_1 be a subset of X and let $X_2 = X - X_1$ such that $|X_1| \leqslant k+1$ and $|X_2| \leqslant 2k-2$. By Corollary 2.4, $G - X_1$ has k-edge-disjointed spanning trees. By Lemma 2.6(iii) and (iv), $F(G_X) \leqslant F((G-X_1)_{X_2}) \leqslant 2$. Using the same argument for the case 1 above, we have $G'_X = K_{2,t}$ $(t \geqslant 2)$.

Therefore, by Lemma 2.7, Theorem 3.1 is proved. □

From the proof of Theorem 3.1, we have the following:

Theorem 3.1′. *Let $r \geqslant 4$ be an integer and let $k = \lfloor \frac{r}{2} \rfloor$. Let G be an r-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leqslant r+k-2$ and let G_X be the graph obtained from G by subdividing every edge in X. Let G'_X be the reduction of G_X. Then exactly one of the following holds*

(i) *G_X is collapsible, or*
(ii) *G_X can be contracted to $K_{2,t}$ (i.e. $G'_X = K_{2,t}$) in such a way that each degree vertex in $K_{2,t}$ is a trivial contraction and $r \leqslant t \leqslant |X|$.*

Theorem 3.2. *Let $r \geqslant 4$ be an integer and let $k = \lfloor \frac{r}{2} \rfloor$. Let G be an r-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leqslant r+k-2$. Then one of the following holds*

(i) *for any even subset $R \subseteq V(G)$, G has a spanning R-trail H_R such that $X \subseteq E(H_R)$, or*
(ii) *X is an edge cut of G and $|X| \geqslant r$.*

Proof. For a given edge set $X \subseteq E(G)$, by Lemma 2.6(ii), if G_X is collapsible, then G has a spanning (R, X)-trail for any even subset $R \subseteq V(G)$. Theorem 3.2 follows from Theorem 3.1. $\square$

Corollary 3.3. *Let $r \geqslant 4$ be an integer, and let $k = \lfloor \frac{r}{2} \rfloor$. Let G be an r-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leqslant r + k - 2$. If X is not an edge cut of G, then G has a spanning (R, X)-trail for any even subset $R \subseteq V(G)$.*

Proof. Following Theorem 3.1 and Lemma 2.6 immediately. $\square$

Corollary 3.4. *Let $r \geqslant 3$. Then G is strongly r-Eulerian-connected if and only if G is $(r+1)$-edge-connected.*

Proof. The necessary condition follows from Theorem 3.0. For the sufficient condition, let $X \subseteq E(G)$ with $|X| \leqslant r$. Then $|X| < \kappa'(G) = r + 1$. X is not an edge cut of G and by Theorem 3.2, the statement holds. $\square$

Theorem 3.5. *Let $r \geqslant 0$. Then*

$$\psi(r) = \theta(r) = \begin{cases} 4 & \text{if } 0 \leqslant r \leqslant 2, \\ r+1 & \text{if } r \geqslant 3. \end{cases}$$

Proof. Since there exist 3-edge-connected graphs which are not supereulerian, $\psi(r) \geqslant \theta(r) \geqslant 4$ for $r \geqslant 0$. By Theorem 3.1, if G is 4-edge-connected, then any edge set X with $|X| \leqslant 2$ can not be an edge cut of G. Therefore G_X is collapsible, and so $\theta(r) = \psi(r) \leqslant 4$ if $r \leqslant 2$. For $r \geqslant 3$, it follows from Corollary 3.4 that $\psi(r) = \theta(r) = r + 1$. $\square$

Corollary 3.6 (*Lai [12]*). *Let $r \geqslant 0$ be an integer. Then*

$$f(r) = \begin{cases} 4, & 0 \leqslant r \leqslant 2, \\ r+1, & r \geqslant 3 \text{ and } r \text{ is odd}, \\ r, & r \geqslant 4 \text{ and } r \text{ is even}. \end{cases}$$

Proof. Since there exist 3-edge-connected graphs that are not supereulerian, $f(r) \geqslant 4$. Since $f(r) \leqslant \theta(r)$, by Theorem 3.1, $f(r) = 4$ if $r \leqslant 2$. For $r \geqslant 3$, if r is odd, Example 1 with an odd number r shows that $f(r) \geqslant r + 1$. By Theorem 3.1, since $f(r) \leqslant \theta(r) \leqslant r + 1$, $f(r) = r + 1$ if r is odd. If r is even, by Theorem 3.1′, for any r-edge-connected graph G and any $X \subseteq E(G)$ with $|X| \leqslant r$, either G_X is collapsible or the reduction $G'_X \cong K_{2,r}$. Since $K_{2,r}$ is supereulerian when r is even and all collapsible graphs are supereulerian, G_X is supereulerian. Then by Lemma 2.6(i), G has a spanning Eulerian subgraph H with $X \subseteq E(H)$. Therefore, $f(r) = r$ if r is even. $\square$

Corollary 3.6 implies that if G is 4-edge-connected, then for any $X \subseteq E(G)$ with $|X| \leqslant 4$, G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$. Here we have:

Theorem 3.7. *Let G be 4-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leqslant 5$. Let G_X be the graph obtained from G by subdividing each edge in X. Let $D_i = \{v \in V(G'_X) \mid d(v) = i\}$ $(i \geqslant 2)$. Then one of the following holds*

(i) *G_X is collapsible, or*
(ii) *X contains an edge cut X_1 with $|X_1| = t \geqslant 4$ such that $G - X_1$ has only two components (H_1 and H_2), which are collapsible. Furthermore, G_X is contractible to $K_{2,t}$ by contracting H_1 and H_2 into the two degree t vertices in $K_{2,t}$, or*
(iii) *G'_X is an Eulerian graph with $V(G'_X) = D_2 \cup D_4$ and $|D_2| = 5$.*

Proof. Let G'_X be the reduction of G_X. If $G'_X = K_1$, then G_X is collapsible and we are done for this case. In the following we will assume that G'_X is not trivial. Since G is 4-edge-connected, G_X is 2-edge-connected. Since $\kappa(G'_X) \geqslant \kappa(G_X)$, G'_X is 2-edge-connected.

Case 1. $F(G'_X) \leqslant 2$.

By Theorem 2.5, and $\kappa'(G_X) \geqslant 2$, $G'_X = K_{2,t}$ for some $t \geqslant 2$. By Lemma 2.7, $|X| \geqslant t \geqslant 4$. Hence, (ii) of Theorem 3.7 holds.

Case 2. $F(G'_X) \geqslant 3$.

Since G is 4-edge-connected and $|X| \leqslant 5$, by (i) and (iii) of Lemma 2.7, $V_r = D_2$ and

$$10 + |D_5| + \cdots + \leqslant 2|V_r| \leqslant 2|X| \leqslant 10.$$

This implies that $|D_i| = 0$ for all $i \geqslant 5$ and $|D_2| = 5$. Therefore, each vertex in $V(G'_X)$ has degree 2 or 4. Hence, G'_X is Eulerian and $|D_2| = 5$. $\square$

Corollary 3.8. *Let G be a 4-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leqslant 5$. Let G_X be the graph obtained from G by subdividing each edge in X. Then either G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$, or G_X is contractible to $K_{2,5}$ in such a way that each path between the two vertices of degree 5 is obtained by subdividing an edge in X.*

Proof. This follows from Theorem 3.7 and Lemma 2.9. $\square$

4. The *r*-edge-Eulerian-connected graphs

We will need the following lemma.

Lemma 4.0. *Let G be a 3-edge-connected graph. Let $X \subseteq E(G)$ and let e', $e'' \in E(G)$. Let $X_0 = X \cup \{e', e''\}$ and let G_{X_0} be the graph obtained from G by subdividing each edge in X_0. Suppose that $G'_{X_0} = K_{2,t}$ where $t \geqslant 3$. If $t > |X|$, then G has a spanning (e', e'')-trail H such that $X \subseteq E(H)$.*

Proof. Let u and v be the two vertices in $K_{2,t}$ with $d(u) = d(v) = t$. By Lemma 2.7, there is an edge set $X_1 \subseteq X_0$ such that each length 2 path between u and v in $K_{2,t}$ is obtained by subdividing an edge in X_1. Then $|X_1| = t$. Let $E_1 = E(G'_{X_0}) = E(K_{2,t})$. By Lemma 2.7, $G_{X_0} - E_1$ has two collapsible subgraphs (H_1 and H_2) such that $V(G_{X_0}) = V(H_1) \cup V(H_2) \bigcup_{e \in X_1} \{v(e)\}$. Let $e' = x'_0 y'_0$, $e'' = x''_0 y''_0$ and let $x'_0, x''_0 \in V(H_1)$ and $y'_0, y''_0 \in V(H_2)$. Since $t > |X|$, at least one of the edges in $\{e', e''\}$ is included in X_1. For each $e \in \{e', e''\}$, P_e is defined as a path obtained by subdividing edge e.

For each H_i, $(i = 1, 2)$, define

$$U_o(H_i) = \{v \in V(H_i) : v \text{ is incident with odd number of edges in } E_1 - \{P_{e'}, P_{e''}\}\}.$$

Note that $|U_o(H_1)|$ is odd if and only if $|U_o(H_2)|$ is odd. Since H_i is collapsible, for any even subset $R_i \subseteq V(H_i)$, there is a spanning connected subgraph Γ_i with $O(\Gamma_i) = R_i$ $(i = 1, 2)$. In the following we will show that a spanning $(v(e'), v(e''))$-trail Γ can be constructed from Γ_1 and Γ_2 by adding all the edges in E_1 and an edge e_{Γ_1} to connect $v(e')$ (or an edge e_{Γ_2} to connect $v(e'')$, or both) such that $O(\Gamma) = \{v(e'), v(e'')\}$.

Case 1. Both e' and e'' are in X_1.

Note that G may not be simple and we may have three possible situations:

(a) $x'_0 = x''_0$ and $y'_0 = y''_0$,
(b) $x'_0 = x''_0$ and $y'_0 \neq y''_0$,
(c) $x'_0 \neq x''_0$ and $y'_0 \neq y''_0$.

The following Tables 1–3 show the selections of the even subset $R_i \subseteq V(H_i)$ for Γ_i and e_{Γ_i} $(i = 1, 2)$ for all possible cases.

For each case with the selection of R_1, R_2, e_{Γ_1} and e_{Γ_2}, define

$$\Gamma = G_{X_0}[E(\Gamma_1) \cup E(\Gamma_2) \cup E_1 \cup \{e_{\Gamma_1}, e_{\Gamma_2}\}].$$

By the definition of Γ, $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2) \bigcup_{e \in X_1} \{v(e)\} \cup \{v(e'), v(e'')\}$, and $v(e')$ and $v(e'')$ have degree 1 in Γ. Since Γ_i is a connected spanning subgraph of H_i, $V(\Gamma_i) = V(H_i)$ $(i = 1, 2)$. Γ_1 and Γ_2 are connected by the paths in E_1, and $v(e')$ and $v(e'')$ are connected to Γ_i by e_{Γ_i}. Thus, $V(\Gamma) = V(G_{X_0})$ and Γ is a connected spanning subgraph

Table 1
When $x'_0 = x''_0$ and $y'_0 = y''_0$, let $x_0 = x'_0 = x''_0$ and $y_0 = y'_0 = y''_0$

$\|U_o(H_1)\|$	x_0 and y_0	R_1	R_2	e_{Γ_1}	e_{Γ_2}
Odd	$x_0 \in U_o(H_1)$, $y_0 \in U_o(H_2)$	$U_o(H_1) - x_0$	$U_o(H_2) - y_0$	$x_0 v(e')$	$v(e'')y_0$
	$x_0 \notin U_o(H_1)$, $y_0 \in U_o(H_2)$	$U_o(H_1) \cup \{x_0\}$	$U_o(H_2) - y_0$	$x_0 v(e')$	$v(e'')y_0$
	$x_0 \in U_o(H_1)$,$y_0 \notin U_o(H_2)$	$U_o(H_1) - x_0$	$U_o(H_2) \cup \{y_0\}$	$x_0 v(e')$	$v(e'')y_0$
	$x_0 \notin U_o(H_1)$, $y_0 \notin U_o(H_2)$	$U_o(H_1) \cup \{x_0\}$	$U_o(H_2) \cup \{y_0\}$	$x_0 v(e')$	$v(e'')y_0$
Even		$U_o(H_1)$	$U_o(H_2)$	$x_0 v(e')$	$x_0 v(e'')$

Table 2
When $x'_0 = x''_0$ and $y'_0 \neq y''_0$, let $x_0 = x'_0 = x''_0$

$\|U_o(H_1)\|$	x_0, and y''_0	R_1	R_2	e_{Γ_1}	e_{Γ_2}
Odd	$x_0 \in U_o(H_1)$, $y''_0 \in U_o(H_2)$	$U_o(H_1) - x_0$	$U_o(H_2) - y''_0$	$x_0 v(e')$	$v(e'')y''_0$
	$x_0 \in U_o(H_1)$, $y''_0 \notin U_o(H_2)$	$U_o(H_1) - x_0$	$U_o(H_2) \cup \{y''_0\}$	$x_0 v(e')$	$v(e'')y''_0$
	$x_0 \notin U_o(H_1)$, $y''_0 \in U_o(H_2)$	$U_o(H_1) \cup \{x_0\}$	$U_o(H_2) - y''_0$	$x_0 v(e')$	$v(e'')y''_0$
	$x_0 \notin U_o(H_1)$, $y''_0 \notin U_o(H_2)$	$U_o(H_1) \cup \{x_0\}$	$U_o(H_2) \cup \{y''_0\}$	$x_0 v(e')$	$v(e'')y''_0$
Even		$U_o(H_1)$	$U_o(H_2)$	$x_0 v(e')$	$x_0 v(e'')$

Table 3
When $x'_0 \neq x''_0$ and $y'_0 \neq y''_0$

$\|U_o(H_1)\|$	x'_0, and y''_0	R_1	R_2	e_{Γ_1}	e_{Γ_2}
Odd	$x'_0 \in U_o(H_1)$, $y''_0 \in U_o(H_2)$	$U_o(H_1) - x'_0$	$U_o(H_2) - y''_0$	$x'_0 v(e')$	$v(e'')y''_0$
	$x'_0 \in U_o(H_1)$, $y''_0 \notin U_o(H_2)$	$U_o(H_1) - x'_0$	$U_o(H_2) \cup \{y''_0\}$	$x'_0 v(e')$	$v(e'')y''_0$
	$x'_0 \notin U_o(H_1)$, $y''_0 \in U_o(H_2)$	$U_o(H_1) \cup \{x'_0\}$	$U_o(H_2) - y''_0$	$x'_0 v(e')$	$v(e'')y''_0$
	$x'_0 \notin U_o(H_1)$, $y''_0 \notin U_o(H_2)$	$U_o(H_1) \cup \{x'_0\}$	$U_o(H_2) \cup \{y''_0\}$	$x'_0 v(e')$	$v(e'')y''_0$
Even	$x'_0 \in U_o(H_1)$, $x''_0 \in U_o(H_1)$	$U_o(H_1) - \{x'_0, x''_0\}$	$U_o(H_2)$	$x'_0 v(e')$	$x''_0 v(e'')$
	$x'_0 \notin U_o(H_1)$, $x''_0 \in U_o(H_1)$	$(U_o(H_1) - \{x''_0\}) \cup \{x'_0\}$	$U_o(H_2)$	$x'_0 v(e')$	$x''_0 v(e'')$
	$x'_0 \in U_o(H_1)$, $x''_0 \notin U_o(H_1)$	$(U_o(H_1) - \{x'_0\}) \cup \{x''_0\}$	$U_o(H_2)$	$x'_0 v(e')$	$x''_0 v(e'')$
	$x'_0 \notin U_o(H_1)$, $x''_0 \notin U_o(H_1)$	$U_o(H_1) \cup \{x'_0, x''_0\}$	$U_o(H_2)$	$x'_0 v(e')$	$x''_0 v(e'')$

of G_{X_0}. To show that $O(\Gamma) = \{v(e'), v(e'')\}$, we can check each case listed in Tables 1–3. For instance, with the cases in Table 1, if $v \notin R_1 \cup R_2$, v has even degree in Γ_1 or Γ_2 or v has degree 2 as a vertex obtained by subdividing an edge in X_1. If $v \in R_1$ and $v \neq x_0$ (or $v \in R_2$ and $v \neq y_0$), then since odd number of edges incident with v in E_1 are added, v has an even degree in Γ. If $v = x_0$ (or y_0), by the definition of e_{Γ_1} and e_{Γ_2}, x_0 has an even degree in Γ. Hence, $O(\Gamma) = \{v(e'), v(e'')\}$, and Γ is a spanning $(v(e'), v(e''))$-trail in G_{X_0}. By Lemma 2.9, G has a spanning (e', e'')-trail containing X.

Case 2. One of e' and e'' is in X_1 (say $e' \in X_1$).

Since $e'' \notin X_1$, we may assume that the path obtained by subdividing e'' is in H_1. Then $v(e'') \in V(H_1)$. For this case, we only need to choose e_{Γ_1} to connect $v(e')$ in Γ.

For each case in Table 4, define

$$\Gamma = G_{X_0}[E(\Gamma_1) \cup E(\Gamma_2) \cup E_1 \cup \{e_{\Gamma_1}\}].$$

Therefore, Γ is a spanning connected subgraph of G_{X_0} such that $O(\Gamma) = \{v(e'), v(e'')\}$. The Lemma is proved. □

In [14], Zhan proved the following:

Theorem 4.1 (*Zhan [14]*). *If G is a 4-edge-connected graph, then for any edges $e_1, e_2 \in E(G)$ there is a spanning (e_1, e_2)-trail in G.*

Table 4
$e' \in X_1$, and $v(e'') \in V(H_1)$

$\lvert U_o(H_1)\rvert$	x_0', and y_0'	R_1	R_2	e_{Γ_1}
Odd	$y_0' \in U_o(H_2)$	$U_o(H_1) \cup \{v(e'')\}$	$U_o(H_2) - y_0'$	$v(e')y_0'$
	$y_0' \notin U_o(H_2)$	$U_o(H_1) \cup \{v(e'')\}$	$U_o(H_2) \cup \{y_0'\}$	$v(e')y_0'$
Even	$x_0' \in U_o(H_1)$	$(U_o(H_1) - \{x_0'\}) \cup \{v(e'')\}$	$U_o(H_2)$	$x_0'v(e')$
	$x_0' \notin U_o(H_1)$	$U_o(H_1) \cup \{x_0', v(e'')\}$	$U_o(H_2)$	$x_0'v(e')$

Theorem 4.1 can be improved.

Theorem 4.2. *Let $r \in \{3, 4\}$. If G is an $(r+1)$-edge-connected graph, then for any $X \subseteq E(G)$ with $|X| \leqslant r-1$, and for any $e_1, e_2 \in E(G)$, G has a spanning (e_1, e_2)-trail H in G such that $X \subseteq E(H)$.*

Proof. Let $X_0 = X \cup \{e_1, e_2\}$. Let G_{X_0} be the graph obtained from G by subdividing each edge in X_0. Since $r \in \{3, 4\}$, $k = \lfloor (r+1)/2 \rfloor = 2$. Then $|X_0| \leqslant |X| + 2 \leqslant r+1 = (r+1)+k-2$. By Theorem 3.1′, either G_{X_0} is collapsible or G_{X_0} is contractible to $K_{2,t}$ with $t \geqslant r$. If G_{X_0} is collapsible, then by Lemma 2.9, G has a spanning (e_1, e_2)-trail containing X. If G_{X_0} is contractible to $K_{2,t}$ with $t \geqslant 4$, since $t \geqslant r > |X|$, by Lemma 4.0, G has a spanning (e_1, e_2)-trail containing the edge set X. $\square$

For graphs with edge-connectivity at least 5, we have

Theorem 4.3. *Let G be an $(r+1)$-edge-connected graph $(r \geqslant 4)$. Let $X \subseteq E(G)$ with $|X| \leqslant r$. Then G is an r-edge-Eulerian-connected.*

Proof. Let e_1 and e_2 be two arbitrary edges in G and let $X_0 = X \cup \{e_1, e_2\}$. Let G_{X_0} be the graph obtained from G by subdividing each edge in X_0.

Case 1. $r \geqslant 5$.

Then $r+1 \geqslant 6$, and so $k = \lfloor (r+1)/2 \rfloor \geqslant 3$. Then $|X_0| \leqslant |X| + 2 \leqslant r+2 \leqslant (r+1)+k-2$. By Theorem 3.1′, either G_{X_0} is collapsible or G_{X_0} is contractible to $K_{2,t}$ with $|X_0| \geqslant t \geqslant (r+1)$. By Lemma 2.9 and Lemma 4.0, both cases imply that G has a spanning (e_1, e_2)-trail H such that $X \subseteq E(H)$. Theorem 4.3 is proved for this case.

Case 2. $r = 4$.

Then G is 5-edge-connected and $|X_0| \leqslant 6$. Let G'_{X_0} be the reduction of G_{X_0}. If $F(G'_{X_0}) \leqslant 2$, then G_{X_0} is either collapsible or contractible to $K_{2,t}$ with $t \geqslant (r+1)$ and so we are done. Next we assume that $F(G'_{X_0}) \geqslant 3$.

Claim. If $v \in D_2 \subseteq V(G'_{X_0})$, then the degree of each of the two neighbors of v is greater than 2.

Since $\delta(G) \geqslant \kappa'(G) \geqslant 5$, each vertex of degree 2 in G'_{X_0} is obtained by subdividing an edge in X_0. If a degree vertex has a neighbor which is also degree , then this will contradict to the definition of G_{X_0}.

By Lemma 2.8, we have

$$|V(G'_{X_0})| + 10 \leqslant 3|D_2| \leqslant 3|X_0|. \tag{8}$$

If $|D_2| \leqslant 5$, then by (8), $|V(G'_{X_0})| \leqslant |D_2| \leqslant 5$, contrary to the claim above. Therefore, $|D_2| = |X_0| = 6$. By (8) and $|D_2| = 6$,

$$|V(G'_{X_0})| \leqslant 8.$$

Therefore, G'_{X_0} is a 2-edge-connected graph with 6 vertices of degree 2 and at most two vertices of degree at least 5. By the claim above, vertices of degree 2 are not adjacent to each other. Therefore, $G'_{X_0} = K_{2,6}$, contrary to $F(G'_{X_0}) \geqslant 3$. The theorem is proved. $\square$

Let r be an integer. Theorem 4.2 shows that if G is 4-edge-connected, then G is 2-edge-Eulerian-connected. If $r \geqslant 4$ and if G is $(r+1)$-edge-connected, then G is r-edge-Eulerian-connected. Combining Theorems 4.2, 4.3 and 3.0, we have:

Corollary 4.4. *Let $r \geqslant 0$ be an integer. Then*

$$\xi(r) = \begin{cases} 4, & 0 \leqslant r \leqslant 2, \\ r+1, & r \geqslant 4. \end{cases}$$

Remark. The case $\xi(3)$ is still open. Theorem 4.2 implies that if G is 5-edge-connected, then G is 3-edge-Eulerian-connected, and so $\xi(3) \leqslant 5$. We conjecture that $\xi(3) = 4$. The following theorem provides some supports for this conjecture.

Theorem 4.5. *Let G be a 4-edge-connected graph and let $X \subseteq E(G)$ with $|X| \leqslant 3$. For any two adjacent edges e' and e'', G has a spanning (e', e'')-trail H such that $X \subseteq E(H)$.*

Proof. Let $X_0 = X \cup \{e', e''\}$. Let G_{X_0} be the graph obtained from G by subdividing each edge in X_0. Let $v(e')$ and $v(e'')$ be the two vertices obtained in the process of subdividing e' and e''. If G_{X_0} is collapsible, then G_{X_0} has a spanning connected subgraph H such that $\mathrm{O}(H) = \{v(e'), v(e'')\}$. By Lemma 2.9, G has a spanning (e', e'')-trail containing X. We are done in this case. Next, we assume that G_{X_0} is not collapsible.

Let G'_{X_0} be the reduction of G_{X_0}. By Theorem 3.7, either $G'_{X_0} = K_{2,t}$ with $t \geqslant 4$ or G'_{X_0} is Eulerian with $V(G_{X_0}) = D_2 \cup D_4$ and $|D_2| = 5$, where D_i is the set of vertices of degree i in G'_{X_0}. If $G'_{X_0} = K_{2,t}$ with $t \geqslant 4$, then by Lemma 4.0, G has a spanning (e', e'')-trail H such that $X \subseteq E(H)$. We are done for this case.

For the case that G'_{X_0} is Eulerian, let v be the vertex incident with both e' and e''. Let $e_1 = v(e')v$ and $e_2 = v(e'')v$. Then $G'_{X_0} - \{e_1, e_2\}$ is connected. Otherwise, $\{e', e''\}$ is an edge cut of G, contrary to that G is 4-edge-connected. Therefore, $G'_{X_0} - \{e_1, e_2\}$ is a connected graph with only two odd degree vertices at $v(e')$ and $v(e'')$. Let $U_4 = \{u \in D_4 : u \text{ is a non-trivial contraction}\}$. For each vertex $u \in U_4$, let $H(u)$ be the preimage of u in G_{X_0}. Then $H(u)$ is collapsible. Let

$$V_u = \{x \in V(H(u)) : x \text{ is incident with odd number of edges in } G'_{X_0} - \{e_1, e_2\}\}.$$

Since $d(u)$ in $G'_{X_0} - \{e_1, e_2\}$ is even, $|V_u|$ is even or 0. Since $H(u)$ is collapsible, $H(u)$ has a spanning connected subgraph Γ_u such that $\mathrm{O}(\Gamma_u) = V_u$. Let $E_0 = E(G_{X_0}) - \{e_1, e_2\}$ and let

$$\Gamma = G_{X_0}\left[\bigcup_{u \in U_4} E(\Gamma_u) \cup E_0\right].$$

Then Γ is a spanning connected subgraph of G_{X_0} such that $\mathrm{O}(\Gamma) = \{v(e'), v(e'')\}$. Therefore, G_{X_0} has a spanning $(v(e'), v(e''))$-trail. By Lemma 2.9, G has a spanning (e', e'')-trail containing X. The proof is complete. □

Acknowledgements

Much of the work for this paper was done while Zhi-Hong Chen was enjoying the hospitality of the JiNan University, Guangzhou, P.R China, which is herewith most gratefully acknowledged.

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